

The iPod Model

Daniel Lanoue

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Abstract

We introduce a Voter Model variant, inspired by social evolution of musical preferences. In our model, agents have preferences over a set of songs and upon meeting update their own preferences incrementally towards those of the other agents they meet. Using the spectral gap of an associated Markov chain, we give a geometry dependent result on the asymptotic consensus time of the model.

1 Introduction

The terminology of Finite Markov Information Exchange (FMIE) models has been introduced [1] [2] as a catch-all for the interpretation of Interacting Particle Systems (IPS) models as stochastic social dynamics. Many important and classical models fit under this two-level framework; the bottom level a meeting model among agents, and the top level an information exchange algorithm performed at each meeting.

For classic IPS models, such as the Voter Model, with a simple meeting algorithm the FMIE perspective is perhaps unnecessary. In this paper however, we will introduce and study a (much) generalized Voter Model - inspired by the evolution of musical preferences among a group of friends - as an FMIE process.

1.1 The iPod Model

Here we introduce the iPod FMIE model. The underlying framework of the stochastic process is a weighted graphs \mathfrak{G} on N vertices. We will refer to each vertex as an agent and occasionally to our vertex set as I . Associated to the edges are symmetric meeting rates $\nu_{i,j}$ for $1 \leq i \neq j \leq N$. We assume that all meeting rates are normalized, i.e.

$$\sum_j \nu_{i,j} = 1$$

for all agents i .

Each agent i is equipped at each time t with a probability measure $X_t(i)$ on $\{1, 2, \dots, \sigma\}$ which we will reference by its distribution $X_t^k(i)$ for $1 \leq k \leq \sigma$.

We consider σ as a fixed number of songs and $X_t^k(i)$ the preference of agent i at time t for song k . The stochastic process X_t updates over time as follows. Between every pair of agents i, j we associate a Poisson process with rate $\nu_{i,j}$ whose times we refer to as meetings between i and j . At a meeting time t

between agents i and j , each agent picks a song σ_i and σ_j independently and distributed according to $X_{t-}(i)$ and $X_{t-}(j)$.

We interpret this as each agent choosing a song to play to the other agent based on their preferences. After agent i hears the song j chose he updates his preferences according to

$$X_t^{\sigma_j}(i) = (1 - \eta)X_{t-}^{\sigma_j}(i) + \eta$$

and

$$X_t^k(i) = (1 - \eta)X_{t-}^k(i)$$

for all other $k \neq \sigma_j$. Here $0 < \eta < 1$ is a fixed interaction parameter. Agent j updates her preferences similarly. It is immediate that if $X_{t-}(i)$ is a probability measure then so is $X_t(i)$. Note that we are implicitly working with cadlag paths.

Analogous to results on the consensus time of the Voter Model - for instance [4] or more generally [8] - in this paper we will estimate the fixation time (to be defined) of the iPod process. Interestingly, again similar to the Voter Model our proof will explore a connection between this process and the Wright-Fisher diffusion [4].

A special feature of the model (Proposition 2.4) is that the average (over agents) preference for a given song evolves as a martingale, analogous to the total proportion of agents with a given opinion on the voter model. This distinguishes the iPod model from many other variants of the voter model that have been studied [3].

1.2 Fixation Time

We will be focused on estimating the fixation time T_{fix} of the iPod process. Every time two agents meet at least one distinct song is played between them and so at least one of the σ songs is played infinitely often. Given that only one song is played infinitely often, we define T_{fix} to be the last time any other song is played.

We note that T_{fix} is not a stopping time and a priori could be infinite, i.e. if more than one song is played infinitely often. However, we will show that this is not the case and in fact T_{fix} has finite expectation, the bounding of which will be our primary goal.

Theorem 1.1. *There exists a constant $C(\eta)$ so that from any initial configuration of σ songs, the fixation time T_{fix} has expectation*

$$\mathbb{E} T_{\text{fix}} \leq C(\eta) \ln(\sigma) \frac{N}{\lambda},$$

where λ is the spectral gap of \mathfrak{G} .

The spectral gap λ of reversible Markov chain is interpreted as its asymptotic rate of convergence to its stationary distribution, and can be defined by the second eigenvalue of the chain's transition matrix [7]. In our setting, we define the spectral gap λ in terms of the edge weights $\nu_{i,j}$. First, for any function $f: I \rightarrow \mathbb{R}$ we define the Dirichlet form $\varepsilon(f, f)$ by

$$\varepsilon(f, f) = \sum_{i,j} \frac{\nu_{i,j}}{2N} (f(i) - f(j))^2.$$

The spectral gap λ is then defined in our context by the extremal characterization

$$\lambda = \inf_{f: I \rightarrow \mathbb{R} \mid \text{Var}(f) \neq 0} \frac{\varepsilon(f, f)}{\text{Var}(f)}.$$

There is extensive literature [7] giving order of magnitude bounds on the $N \rightarrow \infty$ asymptotic behaviour of λ_N for particular families of N -vertex graphs. For such families, Theorem 1.1 gives an order of magnitude upper bound on the asymptotic fixation time, for fixed η and σ . We will show (Theorem 5.1) the tightness of this bound in the case of particular special family of graphs.

2 Projection on a Single Song

Our main technique will be focusing on the projection of our system to a single song. For some fixed (arbitrary) song k we will consider only $X^k(i)$ which we will write simply as $x(i)$ dropping the k . When two agents i, j meet, each independently chooses to either play song k or not; with probability $x(i)$ and $x(j)$ respectively. Writing $\text{Ber}(x(i))$ and $\text{Ber}(x(j))$ for independent Bernoulli variables with given success parameters, we see that if i and j meet at time t that

$$x_t(i) = (1 - \eta)x_{t-}(i) + \eta \text{Ber}(x_{t-}(j)),$$

with $x(j)$ updating similarly. At such a meeting, for all other agents $k \neq i, j$, $x(k)$ remains unchanged.

This implies that the evolution of any given song can be considered separately from the others - though not independently. We will therefore focus first on the FMIE system $\{x_t(i)\}_{i \in I, t \geq 0}$ evolving as above and then later return to the original multi-song model. The primary object of study in our one song model will be the average preference for the song, written

$$M_t = \sum_{i \in I} \frac{x_t(i)}{N}.$$

Our goal in this section will be bounding how long it takes M_t to approach the boundary $\{0, 1\}$. Specifically we will prove a bound on the stopping time

$$S = \inf\{t \geq 0: M_t \notin (\frac{\eta}{2N}, 1 - \frac{\eta}{2N})\}.$$

We will use the shorthand $x_t = \{x_t(i) : 1 \leq i \leq N\}$ for the configuration at time t . In particular, we will often use x_0 for an arbitrary initial configuration. By comparison, we will use X_t (respectively X_0) for a configuration of the multi-song model.

To state our bound, we introduce the function $\phi(x)$ given by

$$\phi(x) = -x \ln(x) - (1 - x) \ln(1 - x). \quad (1)$$

Theorem 2.1. *There exists a constant $A(\eta)$ so that from any initial configuration x_0*

$$\mathbb{E}_{x_0} S \leq A(\eta) \frac{N}{\lambda} \phi(M_0),$$

where λ is the spectral gap of \mathfrak{G} .

To prove our theorem, we first estimate how long it takes M_t to exit small intervals. Then, we use embedding to compare M_t to the Wright-Fisher Diffusion.

2.1 Derived Quantities

We will begin by analysing a few quantities derived from x_t . For ease of notation we will occasionally drop t . Our primary object of study will be the (L^1) average of the preferences $x(i)$, denoted M_t which is introduced above. We will repeatedly make use of the following lemma on the step sizes of M_t .

Lemma 2.2. *If t is a meeting time then*

$$|M_t - M_{t-}| \leq \frac{2\eta}{N}.$$

Proof. If agent i is involved in a meeting at t , then either

$$x_t(i) = (1 - \eta)x_{t-}(i) \text{ or } x_t(i) = (1 - \eta)x_{t-}(i) + \eta,$$

and so

$$|x_t(i) - x_{t-}(i)| \leq \eta.$$

As only two agents are involved in any meeting, our bound follows easily. \square

As a warm-up for the more complicated quantities to appear later, we begin by showing that M_t evolves as a continuous time martingale. We here implicitly use the filtration \mathfrak{F}_t generated by $\{x_t(i)\}_{i \in I, t \geq 0}$. Also, note that we may clearly assume that almost surely meeting times between agents are unique and that the set of meeting times has no accumulation point.

We will make use of the process dynamics notation

$$\mathbb{E}(dA_t | \mathfrak{F}_{t-}) = (\text{resp. } \geq, \leq) B_t dt$$

to mean that

$$A_t - A_0 - \int_0^t B_r dr$$

is a martingale (respectively submartingale, supermartingale). Clearly this notation is compatible with arithmetic operations. To calculate a process's dynamics, we make repeated use of the following lemma, the proof of which is straightforward.

Lemma 2.3. *Let A_t be a function of the $x_t(i)$. Then*

$$\mathbb{E}(dA_t | \mathfrak{F}_{t-}) = \sum_{i,j} \nu_{i,j} \mathbb{E}(A_t - A_{t-} | i \text{ and } j \text{ meet at } t) dt$$

In particular, for the average preference M_t we have the following dynamics.

Proposition 2.4. *With respect to the filtration \mathfrak{F}_t , M_t is a continuous time martingale.*

Proof. To begin we note that since $\mathbb{E} \text{Ber}(x_t(j)) = x_t(j)$ we have that

$$\mathbb{E}(x_t(i)|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}) = (1 - \eta)x_{t-}(i) + \eta x_{t-}(j),$$

and similarly for $x_t(j)$. Summing both we find that

$$\mathbb{E}(x_t(i) + x_t(j)|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}) = x_{t-}(i) + x_{t-}(j).$$

As only $x(i)$ and $x(j)$ change at such a time t , this gives us that

$$\mathbb{E}(M_t|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}) = M_{t-},$$

which clearly implies that

$$\mathbb{E}(dM_t|\mathfrak{F}_{t-}) = 0,$$

i.e. M_t is a martingale. \square

We next look at the process dynamics of M_t^2 . To do so we introduce the quantity Q_t given by

$$Q_t = \sum_{i \in I} \frac{x_t(i)(1 - x_t(i))}{N}.$$

In particular we use Lemma 2.3 to calculate the following.

Proposition 2.5. *The variation M_t^2 satisfies*

$$\mathbb{E}(dM_t^2|\mathfrak{F}_{t-}) = \frac{2\eta^2}{N} Q_t dt.$$

Proof. As before, we begin by calculating that for $k \neq i, j$, since $x(k)$ does not change after a meeting between i and j that:

$$\mathbb{E}(x_t(k)(x_t(i) + x_t(j))|i \text{ and } j \text{ meet at time } t, \mathfrak{F}_{t-}) = x_{t-}(k)(x_{t-}(i) + x_{t-}(j)).$$

Next we calculate that

$$\begin{aligned} \mathbb{E}(x_t^2(i)|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-}) \\ = (1 - \eta)^2 x_{t-}^2(i) + 2\eta(1 - \eta)x_{t-}(i)x_{t-}(j) + \eta^2 x_{t-}^2(j), \end{aligned}$$

and similarly for $x^2(j)$. Finally we have that

$$\begin{aligned} \mathbb{E}(x_t(i)x_t(j)|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-}) \\ = (1 - \eta)^2 x_{t-}(i)x_{t-}(j) + \eta(1 - \eta)[x_{t-}^2(i) + x_{t-}^2(j)] + \eta^2 x_{t-}(i)x_{t-}(j). \end{aligned}$$

Putting this all together we find that

$$\begin{aligned} \mathbb{E}((\sum_i x_t(i))^2|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-}) \\ = (\sum_i x_{t-}(i))^2 + \eta^2(x_{t-}(i) - x_{t-}^2(i) + x_{t-}(j) - x_{t-}^2(j)). \end{aligned}$$

Using Lemma 2.3, summing over i, j and normalizing by N^2 we find that

$$\mathbb{E}(dM_t^2|\mathfrak{F}_{t-}) = \frac{2\eta^2}{N} Q_t dt.$$

\square

Instead of M^2 , we will often be more concerned with $M_t(1 - M_t)$. As M_t is a martingale, from Proposition 2.5 we easily have that

$$\mathbb{E}(dM_t(1 - M_t)|\mathfrak{F}_{t-}) = -\frac{2\eta^2}{N}Q_t dt.$$

A central tool for the study of the underlying Markov Chain on \mathfrak{G} is the Dirichlet form ε . We recall that the Dirichlet form $\varepsilon(f, f)$ for a function $f: I \rightarrow \mathbb{R}$ is defined as

$$\varepsilon(f, f) = \sum_{i,j} \frac{\nu_{ij}}{2N} (f(i) - f(j))^2.$$

We will write $\varepsilon(x_t, x_t)$ for the Dirichlet form of the function $i \mapsto x_t(i)$.

The main fact that we will need about the Dirichlet form is its relationship to the spectral gap. We recall the definition of the spectral gap of a Markov Chain is given by

$$\lambda = \inf_{f: I \rightarrow \mathbb{R} \mid \text{Var}(f) \neq 0} \frac{\varepsilon(f, f)}{\text{Var}(f)},$$

where $\text{Var}(f)$ is the variance of the function $f(i)$ with respect to the uniform measure on I . A simple but important fact we make repeated use of is that $0 < \lambda \leq 1$.

Following Lemma 2.3 we can calculate dQ .

Proposition 2.6. *The sum Q_t satisfies*

$$\mathbb{E}(dQ_t|\mathfrak{F}_{t-}) = 4\eta(1 - \eta)\varepsilon(x_t, x_t)dt - 2\eta^2 Q_t dt,$$

as well as

$$\mathbb{E}(dQ_t|\mathfrak{F}_t) \geq 4\lambda\eta(1 - \eta)M_t(1 - M_t)dt - (2\eta^2 + 4\lambda\eta(1 - \eta))Q_t dt.$$

Proof. We begin by noting that $Q_t = M_t - \sum_i \frac{x_t^2(i)}{N}$ and so

$$\mathbb{E}(dQ_t|\mathfrak{F}_{t-}) = \mathbb{E}\left(d\left(\sum_i \frac{x_t^2(i)}{N}\right)|\mathfrak{F}_{t-}\right)$$

We have from Proposition 2.5 that

$$\begin{aligned} \mathbb{E}(x_t^2(i)|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-}) \\ = (1 - \eta)^2 x_{t-}^2(i) + 2\eta(1 - \eta)x_{t-}(i)x_{t-}(j) + \eta^2 x_{t-}(j). \end{aligned}$$

When agents i and j meet, only $x(i)$ and $x(j)$ change and so

$$\begin{aligned} \mathbb{E}(Q_t - Q_{t-}|i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-}) \\ = -\mathbb{E}\left(\frac{x_t^2(i) - x_{t-}^2(i)}{N} + \frac{x_t^2(j) - x_{t-}^2(j)}{N} | i \text{ and } j \text{ meet at } t, \mathfrak{F}_{t-}\right) \\ = (2\eta - \eta^2)\frac{x_{t-}^2(i) + x_{t-}^2(j)}{N} - 4\eta(1 - \eta)\frac{x_{t-}(i)x_{t-}(j)}{N} \\ - \eta^2\frac{x_{t-}(j) + x_{t-}(i)}{N} \\ = \frac{4\eta(1 - \eta)}{2N}(x_{t-}(i) - x_{t-}(j))^2 \\ - \frac{\eta^2}{N}(x_{t-}(i)(1 - x_{t-}(i)) + x_{t-}(j)(1 - x_{t-}(j))). \end{aligned}$$

Summing over i and j our first equation for dQ_t is done. The second is an immediate consequence of the first using the identity

$$\varepsilon(x, x)_t \geq \lambda \text{Var}(x)_t = \lambda(M_t(1 - M_t) - Q_t).$$

□

2.2 Escaping an ϵ Neighbourhood

Next we focus our attention on how long it takes M_t to escape from the neighbourhood $(M_0 - \epsilon, M_0 + \epsilon)$ for some small (unspecified for now) ϵ . Let τ be the escape time, i.e.

$$\tau = \inf\{t \geq 0: M_t \notin (M_0 - \epsilon, M_0 + \epsilon)\}.$$

For ease of notation in this section we will often write \mathbb{E} for \mathbb{E}_{x_0} - that is the expectation starting from some initial condition x_0 , perhaps with some (to be specified) condition on M_0 .

Our main goal in this section will be to show the following bound.

Proposition 2.7. *There exists a positive constant $A(\eta)$ so that for any M_0 and ϵ satisfying*

$$\frac{\eta}{2N} \leq \epsilon \leq \frac{1}{2}M_0(1 - M_0)$$

the first exit time τ satisfies

$$\mathbb{E} \tau \leq A(\eta) \frac{N}{\lambda} M_0(1 - M_0).$$

2.2.1 Lower Bound for $\mathbb{E} M_\tau^2 - M_0^2$

First we look for a bound on the heterozygosity $M_t(1 - M_t)$. We will make repeated use of the following calculus exercise.

Lemma 2.8. *For a fixed x_0 , if*

$$\epsilon \leq \frac{x_0(1 - x_0)}{2}$$

and $x_0 - \epsilon \leq x \leq x_0 + \epsilon$ then

$$x(1 - x) \geq \frac{1}{2}x_0(1 - x_0).$$

Using our process dynamics calculations we may now begin to bound τ .

Lemma 2.9. *There exist positive constants $C(\eta), D(\eta)$ so that*

$$\mathbb{E} \int_0^\tau Q_r dr \geq C(\eta) \lambda M_0(1 - M_0) \mathbb{E} \tau - D(\eta) (\mathbb{E} Q_\tau - Q_0).$$

Proof. First we recall that from Proposition 2.6 we have a submartingale

$$Y_t = Q_t - Q_0 - 4\lambda\eta(1-\eta) \int_0^t M_r(1-M_r)dr + (2\eta^2 + 4\lambda\eta(1-\eta)) \int_0^t Q_r dr.$$

The Optional Stopping Theorem shows $\mathbb{E} Y_\tau \geq \mathbb{E} Y_0 = 0$, so

$$\begin{aligned} & \mathbb{E} Q_\tau - Q_0 + (2\eta^2 + 4\lambda\eta(1-\eta)) \int_0^\tau Q_r dr \\ & \geq 4\lambda\eta(1-\eta) \mathbb{E} \int_0^\tau M_r(1-M_r)dr \\ & \geq 4\lambda\eta(1-\eta) \mathbb{E} \int_0^\tau \frac{1}{2} M_0(1-M_0)dr \text{ by Lemma 2.8} \\ & \geq 2\lambda\eta(1-\eta)M_0(1-M_0) \mathbb{E} \tau. \end{aligned}$$

Next, we note that since $\lambda \leq 1$

$$2\eta^2 + 4\lambda\eta(1-\eta) \leq 4\eta - 2\eta^2.$$

Substituting this in and rearranging the inequality

$$(4\eta - 2\eta^2) \mathbb{E} \int_0^\tau Q_r dr \geq 2\lambda\eta(1-\eta)M_0(1-M_0) \mathbb{E} \tau - \mathbb{E} Q_\tau + Q_0$$

and so

$$\mathbb{E} \int_0^\tau Q_r dr \geq C(\eta)\lambda M_0(1-M_0) \mathbb{E} \tau - D(\eta)(\mathbb{E} Q_\tau - Q_0)$$

for $C(\eta) = \frac{2\eta(1-\eta)}{4\eta-2\eta^2}$ and $D(\eta) = \frac{1}{4\eta-2\eta^2}$. □

Using this we are ready for our lower bound.

Lemma 2.10. *There exist positive constants $A(\eta), B(\eta)$ with*

$$\mathbb{E} M_\tau^2 - M_0^2 \geq \frac{1}{N} (A(\eta)\lambda M_0(1-M_0) \mathbb{E} \tau - B(\eta) (\mathbb{E} Q_\tau - \mathbb{E} Q_0)).$$

Proof. Proposition 2.5 shows

$$M_t^2 - M_0^2 - \frac{2\eta^2}{N} \int_0^t Q_r dr$$

is a martingale. The Optional Stopping Theorem and Lemma 2.9 show

$$\mathbb{E} M_\tau^2 - M_0^2 \geq \frac{2\eta^2}{N} (C(\eta)\lambda M_0(1-M_0) \mathbb{E} \tau - D(\eta)(\mathbb{E} Q_\tau - Q_0)),$$

which finishes our proof. □

2.2.2 Upper Bound for $\mathbb{E} M_\tau^2 - M_0^2$

We will now impose the condition on our ϵ neighbourhoods that

$$\frac{\Delta}{4} \leq \epsilon \leq \frac{M_0(1 - M_0)}{2},$$

where $\Delta = \frac{2\eta}{N}$ is the maximum step size for M_t .

Lemma 2.11. *For τ the first escape time from $(M_0 - \epsilon, M_0 + \epsilon)$ we have*

$$\mathbb{E} M_\tau^2 - M_0^2 \leq \frac{25}{4} M_0^2 (1 - M_0)^2$$

Proof. Consider the martingale $Y_t = M_t - M_0$. From the bound on step sizes, we have that at τ

$$|Y_\tau| \leq \epsilon + \Delta \leq 5\epsilon$$

and so

$$\mathbb{E} Y_\tau^2 \leq 25\epsilon^2 \leq \frac{25}{4} M_0^2 (1 - M_0)^2.$$

From the Optional Stopping Theorem

$$\mathbb{E} Y_\tau^2 = \mathbb{E}(M_\tau - M_0)^2 = \mathbb{E} M_\tau^2 - M_0^2.$$

□

2.2.3 Bounding $\mathbb{E} \tau$

We can now prove Proposition 2.7.

Proof. Combining Lemma 2.10 and Lemma 2.11 in opposite ways, we have

$$\frac{25}{4} M_0^2 (1 - M_0)^2 \geq \frac{1}{N} (A(\eta) \lambda M_0 (1 - M_0) \mathbb{E} \tau - B(\eta) (\mathbb{E} Q_\tau - \mathbb{E} Q_0)).$$

Rearranging this and redefining the constants $A(\eta)$ and $B(\eta)$ we have that

$$\mathbb{E} \tau \leq A(\eta) \frac{N}{\lambda} M_0 (1 - M_0) + \frac{B(\eta)}{\lambda M_0 (1 - M_0)} (\mathbb{E} Q_\tau - \mathbb{E} Q_0).$$

Noting that by assumption $M_0(1 - M_0) \geq \frac{\eta}{N}$ and so $\frac{1}{M_0(1 - M_0)} \leq \frac{N}{\eta^2}$, we find

$$\mathbb{E} \tau \leq \frac{N}{\lambda} \left(A(\eta) M_0 (1 - M_0) + \frac{B(\eta)}{\eta^2} (\mathbb{E} Q_\tau - \mathbb{E} Q_0) \right).$$

Finally, since $M_t(1 - M_t)$ is a supermartingale the Optional Stopping Theorem gives $\mathbb{E} M_\tau(1 - M_\tau) \leq M_0(1 - M_0)$. Jensen's inequality implies $Q_t \leq M_t(1 - M_t)$ always and so combining these two facts

$$\mathbb{E} Q_\tau - Q_0 \leq \mathbb{E} M_\tau(1 - M_\tau) \leq M_0(1 - M_0),$$

giving that

$$\mathbb{E} \tau \leq \frac{N}{\lambda} M_0 (1 - M_0) \left(A(\eta) + \frac{B(\eta)}{\eta^2} \right)$$

which completes our proof. □

2.3 Embedding in the Wright-Fisher Diffusion

We will now analyse the martingale M_t , starting from some initial configuration x_0 , by discretizing it and embedding that into a Brownian motion. We begin by defining a series of stopping times τ_k for the martingale M_t . Let $\tau_0 = 0$ and for $k \geq 1$ define τ_k inductively as

$$\tau_k = \inf \left\{ t \geq \tau_{k-1} : |M_t - M_{\tau_{k-1}}| \geq \frac{M_{\tau_{k-1}}(1 - M_{\tau_{k-1}})}{2} \right\},$$

that is the first time after τ_{k-1} that M_t exits the ball of size $\frac{M_{\tau_{k-1}}(1 - M_{\tau_{k-1}})}{2}$ around $M_{\tau_{k-1}}$.

Using Proposition 2.7, we have the following bound on the expectation of the increments of our stopping times.

Lemma 2.12. *There exists a constant $A(\eta)$ so that, from any initial x_0 , assuming that $M_{\tau_{k-1}} \in (\Delta, 1 - \Delta)$:*

$$\mathbb{E}(\tau_k - \tau_{k-1} | \mathcal{F}_{\tau_{k-1}}) \leq A(\eta) \frac{N}{\lambda} M_{\tau_{k-1}}(1 - M_{\tau_{k-1}}).$$

Proof. This follows immediately from Proposition 2.7, applying the Strong Markov property at time τ_{k-1} . \square

2.3.1 The Wright-Fisher Diffusion

Here we introduce the Wright-Fisher diffusion W_t . We recall that W_t is the continuous martingale in $[0, 1]$ solving the stochastic differential equation

$$dW_t = \sqrt{W_t(1 - W_t)} dB_t.$$

We will frequently think of W_t as a diffusion process and so record here that W_t has instantaneous drift $\mu(x) = 0$ and variance $\sigma^2(x) = x(1 - x)$.

Lemma 2.13. *From any initial $W_0 = w_0$, if $\epsilon \leq \frac{w_0(1 - w_0)}{2}$ then the first escape τ from the ϵ -ball about w_0 satisfies*

$$\frac{1}{3}w_0(1 - w_0) \leq \mathbb{E}_{w_0} \tau \leq \frac{5}{3}w_0(1 - w_0).$$

Proof. Let $w_{\pm} = w_0 \pm \epsilon$. On the interval $[w_-, w_+]$ let

$$u(x) = \mathbb{E}_x \tau.$$

Applying a standard argument for diffusion processes we find that $u(x)$ satisfies the equation

$$-1 = \frac{x(1 - x)}{2} u''(x),$$

subject to the boundary conditions $u(w_-) = u(w_+) = 0$. Integrating, we find that for

$$f(x) = 2(x \ln x + (1 - x) \ln(1 - x))$$

we have the solution

$$u(x) = -f(x) + Ax + B$$

for some constants A, B . Applying the boundary conditions we get

$$2\epsilon A = f(w_+) - f(w_-)$$

and

$$B = f(w_+) - Aw_+.$$

Therefore, for w_0 we have

$$\begin{aligned} u(w_0) &= -f(w_0) + Aw_0 + f(w_+) - Aw_+ \\ &= -f(w_0) - A\epsilon + f(w_+) \\ &= -f(w_0) - \frac{f(w_+) - f(w_-)}{2} + f(w_+) \\ &= -f(w_0) + \frac{1}{2}(f(w_+) + f(w_-)). \end{aligned}$$

To simplify this we apply the Taylor approximation to f at w_0

$$f(w_0 + \Delta) \approx f(w_0) + f'(w_0)\Delta + \frac{1}{w_0(1-w_0)}\Delta^2,$$

which plugging into $u(w_0)$ we find

$$u(w_0) \approx \frac{\epsilon^2}{w_0(1-w_0)}$$

To approximate the error, we use Taylor's remainder theorem. First,

$$f^{(3)}(x) = \frac{2}{(1-x)^2} - \frac{2}{x^2} = \frac{4x-2}{x^2(1-x)^2}$$

and so

$$|f^{(3)}(x)| \leq \frac{2}{x^2(1-x)^2}.$$

On the interval $[w_0 - \epsilon, w_0 + \epsilon]$, applying a bound we used for $M_t(1-M_t)$ above, we have

$$x(1-x) \geq \frac{w_0(1-w_0)}{2}.$$

Therefore the third derivative is bounded on the same interval by

$$|f^{(3)}(x)| \leq \frac{8}{w_0^2(1-w_0)^2}.$$

Therefore the error $R(w_0 + x)$ for $|x| \leq \epsilon$ of the 2nd Taylor approximation is bounded by

$$|R(w_0 + x)| \leq \frac{8}{w_0^2(1-w_0)^2} \frac{|x|^3}{6}.$$

So, for w_{\pm} , using that $\epsilon \leq \frac{w_0(1-w_0)}{2}$ we have

$$|R(w_{\pm})| \leq \frac{2\epsilon^2}{3w_0(1-w_0)}.$$

Applying this to our calculation of $u(w_0)$ - once to each of $f(w_0 \pm \epsilon)$ we have

$$\begin{aligned} \left| u(w_0) - \frac{\epsilon^2}{w_0(1-w_0)} \right| &\leq \frac{1}{2} |R(w_0 - \epsilon) + R(w_0 + \epsilon)| \\ &\leq \frac{2\epsilon^2}{3w_0(1-w_0)}, \end{aligned}$$

from which our bounds on $u(w_0)$ follow easily. \square

We will also need the well known bound for the absorption time T_{abs} of W_t at the boundary $\{0, 1\}$. To state this bound, recall the function

$$\phi(x) = -x \ln(x) - (1-x) \ln(1-x). \quad (2)$$

Lemma 2.14. *Let T_{abs} be the stopping time*

$$T_{\text{abs}} = \inf\{t \geq 0: W_t \notin (0, 1)\},$$

i.e. the time when W_t is absorbing in $\{0, 1\}$. Then $\mathbb{E}_x T_{\text{abs}} = 2\phi(x)$ for any $x \in [0, 1]$.

Proof. This is easy using the same techniques as the previous proof. Letting

$$u(x) = \mathbb{E}_x T_{\text{abs}}$$

we find that

$$u(x) = -2(x \ln x + (1-x) \ln(1-x)) + Ax + B,$$

subject to the boundary conditions $u(0) = u(1) = 0$. This necessitates $A = B = 0$ and so

$$u(x) = -2(x \ln x + (1-x) \ln(1-x)),$$

completing the proof. \square

2.3.2 Embedding M_{τ_k} in W_t

Let W_t be a Wright-Fisher diffusion started at $W_0 = M_0$. As $|M_t| \leq 1$ the discrete martingale $\{M_{\tau_k}\}_{k \geq 0}$ is clearly square integrable and so [5] we can find a sequence of stopping times $\tilde{\tau}_k$ for W_t so that

$$\{M_{\tau_k}\}_{k \geq 0} \stackrel{d}{=} \{W_{\tilde{\tau}_k}\}_{k \geq 0}. \quad (3)$$

We will use this embedding to bound the first escape time S by comparison with the absorption time of the Wright-Fisher diffusion.

2.3.3 The Comparison Calculation

We will focus on the first time that M_t exits the interval $(\frac{\eta}{2N}, 1 - \frac{\eta}{2N})$. Recall the stopping time S defined by

$$S = \inf\{t \geq 0: M_t \notin (\frac{\eta}{2N}, 1 - \frac{\eta}{2N})\}, \quad (4)$$

and let

$$K = \inf\{k \geq 0: \tau_k \geq S\}.$$

Lemma 2.15. $K < \infty$ almost surely.

Proof. For any $0 \leq x \leq \frac{1}{2}$ we have

$$x - \frac{x(1-x)}{2} \leq \frac{3}{4}x,$$

and so if $M_0 \leq \frac{1}{2}$, then for any $k \geq \frac{\ln(\frac{\eta}{N})}{\ln(3/4)}$ we have $M_{\tau_k} \leq \frac{\eta}{2N}$ with positive probability. Fix such a k_0 . In fact, as $\{M_{\tau_n}\}_{n \geq 0}$ is a martingale, we have

$$\mathbb{P}\left(M_{\tau_{k_0}} \leq \frac{\eta}{2N}\right) \geq \frac{1}{2^{k_0}}.$$

A similar argument holds for an initial configuration with $\frac{1}{2} \leq M_0 \leq 1$ going above the level $1 - \frac{\eta}{2N}$. Therefore, for any initial M_0 , there is a uniform lower bound on the probability that $K < k_0$. Thus, by a standard Strong Markov argument, we must have $K < \infty$ almost surely. \square

Next, we define equivalent stopping times for W_t . Let \tilde{K} be the index

$$\tilde{K} = \inf\{k \geq 0 : W_{\tilde{\tau}_k} \notin (\frac{\eta}{2N}, 1 - \frac{\eta}{2N})\}$$

and recall the absorption time T_{abs} given by

$$T_{\text{abs}} = \inf\{t \geq 0 : W_t \notin (0, 1)\}.$$

For M_t we have clearly that

$$S \leq \tau_K. \tag{5}$$

Furthermore, as $M_{\tau_K} \in (0, 1)$ so must be $W_{\tilde{\tau}_K}$ by the equivalence in distribution (and thus support). Therefore,

$$\tilde{\tau}_{\tilde{K}} \leq T_{\text{abs}},$$

as for $t \geq T_{\text{abs}}$, W_t is constant and in $\{0, 1\}$.

First we need to bound the $\tilde{\tau}_k$.

Lemma 2.16. *The hitting times $\tilde{\tau}_k$ satisfies*

$$\mathbb{E}(\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}) \geq \frac{1}{3} W_{\tilde{\tau}_{k-1}} (1 - W_{\tilde{\tau}_{k-1}}).$$

Proof. For this, we note that starting at $w_0 = W_{\tilde{\tau}_{k-1}}$, the time $\tilde{\tau}_k$ can only occur after W_t leaves the interval $(W_{\tilde{\tau}_k} - \frac{w_0(1-w_0)}{2}, W_{\tilde{\tau}_k} + \frac{w_0(1-w_0)}{2})$ as $W_{\tilde{\tau}_k}$ is already outside this interval and W_t is continuous. Write τ for the first exit time of this interval. Applying the Strong Markov Property we see that

$$\begin{aligned} \mathbb{E}(\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}) &\geq \mathbb{E}(\tau | F_{\tilde{\tau}_{k-1}}) \\ &\geq \frac{1}{3} W_{\tilde{\tau}_{k-1}} (1 - W_{\tilde{\tau}_{k-1}}) \text{ by Lemma 2.13,} \end{aligned}$$

completing our proof. \square

We are now ready to prove *Theorem 2.1*.

Proof. We recall by Equation (5), $\mathbb{E} S \leq \mathbb{E} \tau_K$ and so we will focus on bounding $\mathbb{E} \tau_K$. As $K < \infty$ almost surely by Lemma 2.15 we have that

$$\mathbb{E} \tau_K = \mathbb{E} \sum_{k=1}^{\infty} (\tau_k - \tau_{k-1}) 1_{K \geq k}.$$

By Lemma 2.12

$$\mathbb{E} (\tau_k - \tau_{k-1} | F_{\tau_{k-1}}) \leq A(\eta) \frac{N}{\lambda} M_{\tau_{k-1}} (1 - M_{\tau_{k-1}}),$$

for some constant $A(\eta)$ depending only on η . Therefore we can calculate using the Strong Markov property that

$$\begin{aligned} \mathbb{E} (\tau_k - \tau_{k-1}) 1_{K \geq k} &= \mathbb{E} \mathbb{E} ((\tau_k - \tau_{k-1}) 1_{K \geq k} | F_{\tau_{k-1}}) \\ &= \mathbb{E} 1_{K \geq k} \mathbb{E}_{x_{\tau_{k-1}}} (\tau_k - \tau_{k-1}) \\ &\leq A(\eta) \frac{N}{\lambda} \mathbb{E} 1_{K \geq k} M_{\tau_{k-1}} (1 - M_{\tau_{k-1}}). \end{aligned}$$

From Equation (3) $\{M_{\tau_k}\}_{k \geq 0}$ and $\{W_{\tilde{\tau}_k}\}_{k \geq 0}$ are equivalent in distribution, so

$$\mathbb{E} 1_{K \geq k} M_{\tau_{k-1}} (1 - M_{\tau_{k-1}}) = \mathbb{E} 1_{\tilde{K} \geq k} W_{\tilde{\tau}_{k-1}} (1 - W_{\tilde{\tau}_{k-1}}).$$

By Lemma 2.13

$$\frac{1}{3} W_{\tilde{\tau}_{k-1}} (1 - W_{\tilde{\tau}_{k-1}}) \leq \mathbb{E} (\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}),$$

so we can calculate

$$\begin{aligned} \mathbb{E} 1_{\tilde{K} \geq k} W_{\tilde{\tau}_{k-1}} (1 - W_{\tilde{\tau}_{k-1}}) &\leq \mathbb{E} 1_{\tilde{K} \geq k} 3 \mathbb{E} (\tilde{\tau}_k - \tilde{\tau}_{k-1} | F_{\tilde{\tau}_{k-1}}) \\ &= 3 \mathbb{E} \mathbb{E} ((\tilde{\tau}_k - \tilde{\tau}_{k-1}) 1_{\tilde{K} \geq k} | F_{\tilde{\tau}_{k-1}}) \\ &= 3 \mathbb{E} (\tilde{\tau}_k - \tilde{\tau}_{k-1}) 1_{\tilde{K} \geq k}. \end{aligned}$$

Therefore we see that

$$\begin{aligned} \mathbb{E} \tau_S &\leq 3 \frac{N}{\lambda} A(\eta) \mathbb{E} \sum_{k \geq 0} \mathbb{E} (\tilde{\tau}_k - \tilde{\tau}_{k-1}) 1_{\tilde{K} \geq k} \\ &\leq 3 \frac{N}{\lambda} A(\eta) \mathbb{E} \tilde{\tau}_{\tilde{K}} \\ &\leq 3 \frac{N}{\lambda} A(\eta) \mathbb{E} T_{\text{abs}} \end{aligned}$$

Because Equation (5) $S \leq \tau_K$ and using Lemma 2.14 to bound $\mathbb{E} T_{\text{abs}}$ we can conclude that

$$\begin{aligned} \mathbb{E} S &\leq 3 \frac{N}{\lambda} A(\eta) \mathbb{E}_{W_0} T_{\text{abs}} \\ &= 6A(\eta) \frac{N}{\lambda} \phi(W_0) \\ &= 6A(\eta) \frac{N}{\lambda} \phi(M_0) \end{aligned}$$

from which our conclusion follows. \square

3 The General Model

We are now ready to prove our bound on the fixation time of the general iPod model with σ songs. We recall that for each agent i , we write their preference for song k by $X_t^k(i)$. For each song k , we write the average preference for that song as M_t^k , given by

$$M_t^k = \sum_i \frac{X_t^k(i)}{N}.$$

We have shown in Proposition 2.4 that for each $1 \leq k \leq \sigma$, M_t^k is a martingale.

3.1 Approaching the Boundary

We begin by using Theorem 2.1 to bound the time it takes for one of the M_t^k to approach the boundary 1. Specifically, we will analyse the stopping time

$$\tau = \inf\{t \geq 0: M_t^k \geq 1 - \frac{\eta}{2N} \text{ for some } 1 \leq k \leq \sigma\}. \quad (6)$$

Let S^k be the first time that M_t^k approaches either boundary, that is

$$S^k = \inf\{t \geq 0: M_t^k \notin (\frac{\eta}{2N}, 1 - \frac{\eta}{2N})\},$$

and set $S_{\max} = \max_{1 \leq k \leq \sigma} S^k$. We will bound τ in two steps, first by bounding $\mathbb{E} S_{\max}$ and second by showing that $\mathbb{E} \tau$ is on the same order of magnitude as $\mathbb{E} S_{\max}$.

Proposition 3.1. *There is a constant $C(\eta)$ so that from any initial configuration X_0 we have*

$$\mathbb{E} S_{\max} \leq C(\eta) \ln(\sigma) \frac{N}{\lambda}.$$

Proof. To begin we recall that by Theorem 2.1 there is a constant $C(\eta)$ so that

$$\mathbb{E} S^k \leq C(\eta) \frac{N}{\lambda} \phi(M_0^k),$$

for all $1 \leq k \leq \sigma$. Clearly $S_{\max} \leq \sum_{k=1}^{\sigma} S^k$ and so

$$\mathbb{E} S_{\max} \leq C(\eta) \frac{N}{\lambda} \sum_{k=1}^{\sigma} \phi(M_0^k).$$

Recalling that the M_0^k satisfy the constraint $\sum_{k=1}^{\sigma} \phi(M_0^k) = 1$, a simple calculus exercise in Lagrange multipliers shows that $\sum_{k=1}^{\sigma} \phi(M_0^k)$ is maximized when $M_0^k = \frac{1}{\sigma}$ for all k and so

$$\begin{aligned} \sum_{k=1}^{\sigma} \phi(M_0^k) &= \sigma \left(-\frac{1}{\sigma} \ln\left(\frac{1}{\sigma}\right) - \left(1 - \frac{1}{\sigma}\right) \ln\left(1 - \frac{1}{\sigma}\right) \right) \\ &\leq \ln(\sigma) + 1, \end{aligned}$$

completing the proof. \square

Next we will show that $\mathbb{E} \tau$ is on the same order of magnitude (w.r.t N) as $\mathbb{E} S_{\max}$.

Proposition 3.2. *From any initial configuration X_0*

$$\mathbb{E} \tau \leq 2 \sup_{X_0} \mathbb{E}_{X_0} S_{\max}.$$

Proof. For $1 \leq k \leq \sigma$ let

$$A_k = \{M_{S_k}^k \geq 1 - \frac{\eta}{2N}\},$$

that is the event that song k approaches the boundary 1 at time S_k (as opposed to the boundary 0). Clearly $A_k \subset \{\tau \leq S_{\max}\}$ and so

$$\mathbb{P}(\tau \leq S_{\max}) \geq \mathbb{P}(\cup_{1 \leq k \leq \sigma} A_k).$$

We claim that the A_k are almost surely disjoint. Assuming otherwise, given $A_k \cap A_j$ one of S^k, S^j must occur first - since both M^k and M^j can't be greater than $1 - \frac{\eta}{2N}$ at the same time. Assume without loss of generality that $S^k < S^j$. At time S^k , $M^k \geq 1 - \frac{\eta}{2N}$ and so we must have $M^j \leq \frac{\eta}{2N}$, meaning that S^j has already occurred, a contradiction. Therefore the events A_k , $1 \leq k \leq \sigma$ are almost surely disjoint.

Next, as M_t^k is a martingale - with step size bounded by $\Delta = \frac{2\eta}{N}$ - we have

$$\mathbb{P}(A_k) \geq \frac{M_0^k - \frac{\eta}{2N}}{1 - \frac{\eta}{2N} + \Delta},$$

and so for $N \gg 0$

$$\mathbb{P}(A_k) \geq \frac{M_0^k}{2}.$$

Thus for any initial configuration X_0

$$\mathbb{P}_{X_0}(\tau \leq S_{\max}) = \sum_k \mathbb{P}(A_k) \geq \sum_k \frac{M_0^k}{2} = \frac{1}{2},$$

or equivalently $\mathbb{P}_{X_0}(\tau \geq S_{\max}) \leq \frac{1}{2}$.

Let $m = \sup_{X_0} \mathbb{E}_{X_0} \tau$. Applying the Strong Markov Property we have

$$\begin{aligned} \mathbb{E}_{X_0} \tau &\leq \mathbb{E}_{X_0} S_{\max} + \mathbb{E}_{X_0} (\tau - S_{\max}) 1_{\tau \geq S_{\max}} \\ &\leq \mathbb{E}_{X_0} S_{\max} + \mathbb{E} 1_{\tau \geq S_{\max}} \mathbb{E}_{X_{S_{\max}}} \tau \\ &\leq \mathbb{E}_{X_0} S_{\max} + \mathbb{P}_{X_{S_{\max}}}(\tau \geq S_{\max}) m \\ &\leq \mathbb{E}_{X_0} S_{\max} + \frac{m}{2} \end{aligned}$$

which implies that

$$m \leq 2 \sup_{X_0} \mathbb{E}_{X_0} S_{\max}.$$

□

3.2 Fixation Time

Next we will estimate the fixation time given that the preference M_t^k for some (fixed but arbitrary) song k has approached the boundary 1. Specifically, we will consider starting from an initial configuration X_0 with

$$M_0^k \geq 1 - \frac{\eta}{2N}.$$

When M^k is near 1, the fixation time T_{fix} can only be the last time any song other than k plays. Of course this need not occur. Projecting on k , this is the last time one of the Bernoulli trials for k has failed. We begin by showing that from such an initial configuration, T_{fix} has likely already occurred.

Proposition 3.3. *From an initial configuration X_0 with $M_0^k \geq 1 - \frac{\eta}{2N}$, we have*

$$\mathbb{P}_{X_0}(T_{\text{fix}} = 0) \geq \frac{1}{2}$$

Proof. We will consider the stopping time R , the first time any song other than k plays. Before R , each $X^k(i)$ can only increase. Therefore at time R - without loss of generality, a meeting of i and j - if another song is played by only one of i, j then

$$X_R^k(i) + x_R^k(j) = (1 - \eta)(X_{R-}(i) + X_{R-}(j)) + \eta \quad (7)$$

$$\leq 2(1 - \eta) + \eta \quad (8)$$

$$= 2 - \eta. \quad (9)$$

If both agents play a different song, then $X^k(i) + X^k(j)$ is even smaller at R .

This then implies that on $\{R < \infty\}$

$$M_R^k \leq 1 - \frac{\eta}{N}.$$

Now, applying the Optional Stopping Theorem to $R \wedge t$, we find that

$$1 - \frac{\eta}{2N} \leq M_0 \quad (10)$$

$$= \mathbb{E} M_{R \wedge t}^k \quad (11)$$

$$= \mathbb{E} (M_R^k 1_{R \leq t} + M_t^k 1_{t < R}) \quad (12)$$

$$\leq (1 - \frac{\eta}{N})(1 - \mathbb{P}(t < R)) + 1 \mathbb{P}(t < R). \quad (13)$$

Solving for $\mathbb{P}(t < R)$ we find that

$$\mathbb{P}(t < R) \geq \frac{1}{2}.$$

As this is true for arbitrary t , we have $\mathbb{P}(R = \infty) \geq \frac{1}{2}$ from which our result follows. \square

Next we need to consider what happens when M_t^k approaches 1, but the song k fails to play at a meeting.

Proposition 3.4. Consider the stopping time R given by

$$R = \inf\{t \geq 0: \text{ some song other than } k \text{ plays at } t\}.$$

From any initial configuration $M_0^k \geq 1 - \frac{\eta}{2N}$, we have

$$\mathbb{E} R 1_{R < \infty} \leq \frac{1}{8\eta}.$$

Proof. Let T_n , $1 \leq n < \infty$ be the n -th meeting time. We first define

$$\tilde{R} = \inf\{n \geq 0: \text{ some song other than } k \text{ plays at } T_n\}.$$

We will calculate how M_t^k changes after the first meeting time, given that song k is played by both agents at the meeting time T_1 .

If agents i and j meet and both play k at T_1 then

$$X_{T_1}^k(i) = (1 - \eta)X_0^k(i) + \eta$$

and similarly for $X^k(j)$. So given that i and j meet and play k

$$M_{T_1}^k = M_0^k - \frac{\eta(X_0^k(i) + X_0^k(j))}{N} + \frac{2\eta}{N}.$$

Summing over pairs of agents we find that

$$\begin{aligned} & \mathbb{E}(M_{T_1}^k | \text{ both agents play } k \text{ at } T_1, \mathfrak{F}_0) \\ &= \sum_{i,j} \mathbb{E}(M_{T_1}^k | i \text{ meets } j, \text{ both play } k \text{ at } T_1, \mathfrak{F}_0) \mathbb{P}(i \text{ meets } j \text{ at } T_1 | \mathfrak{F}_0) \\ &= \sum_{i,j} \frac{\nu_{ij}}{N} \mathbb{E}\left(M_0^k - \frac{\eta(X_0^k(i) + X_0^k(j) - 2)}{N} | i \text{ \& } j \text{ both play } k \text{ at } T_1, \mathfrak{F}_0\right) \\ &= \sum_{i,j} \frac{\nu_{ij}}{N} (M_0^k - \frac{\eta(X_0^k(i) + X_0^k(j) - 2)}{N}) \\ &= M_0^k + \frac{2\eta}{N} - \sum_{i,j} \frac{\nu_{ij}}{N} \frac{\eta(X_0^k(i) + X_0^k(j))}{N} \\ &= M_0^k + \frac{2\eta}{N} - \frac{2\eta M_0^k}{N} \\ &= (1 - \frac{2\eta}{N})M_0^k + \frac{2\eta}{N}. \end{aligned}$$

By the same calculation we find that

$$\mathbb{E}(M_{T_2}^k | \text{ both agents play } k \text{ at } T_2, \mathfrak{F}_{T_1}) = (1 - \frac{2\eta}{N})M_{T_1}^k + \frac{2\eta}{N}$$

and so

$$\begin{aligned} & \mathbb{E}(M_{T_2}^k | \text{ both agents play } k \text{ at } T_1 \text{ and } T_2, \mathfrak{F}_0) \\ &= (1 - \frac{2\eta}{N}) \left((1 - \frac{2\eta}{N})M_0^k + \frac{2\eta}{N} \right) + \frac{2\eta}{N} \\ &= (1 - \frac{2\eta}{N})^2 M_0^k + 1 - (1 - \frac{2\eta}{N})^2 \\ &= 1 - (1 - \frac{2\eta}{N})^2 (1 - M_0^k). \end{aligned}$$

Continuing the same easy inductive calculation we find that

$$\mathbb{E} \left(M_{T_n}^k | \tilde{S} > n, \mathfrak{F}_0 \right) = 1 - \left(1 - \frac{2\eta}{N} \right)^n (1 - M_0^k).$$

Next, we need to know the chance of some song other than k being played at time T_n given $M_{T_{n-1}}^k$. We will need the identity

$$1 - xy \leq (1 - x) + (1 - y)$$

for $x, y \leq 1$ - which follows easily from $1 + (1 - x)(1 - y) \geq 1$. Using that, and that the probability of at least one of i, j not playing k is $1 - X^k(i)X^k(j)$, we have

$$\begin{aligned} & \mathbb{P} \left(\text{A song other than } k \text{ is played at } T_n | M_{T_{n-1}}^k \right) \\ &= \sum_{i,j} \frac{\nu_{ij}}{N} \mathbb{P} \left(\text{Another song is played at } T_n | M_{T_{n-1}}^k, i \text{ meets } j \text{ at } T_n \right) \\ &= \sum_{i,j} \frac{\nu_{ij}}{N} \left(1 - X_{T_{n-1}}^k(i) X_{T_{n-1}}^k(j) \right) \\ &\leq \sum_{i,j} \frac{\nu_{ij}}{N} \left(1 - X_{T_{n-1}}^k(i) + 1 - X_{T_{n-1}}^k(j) \right) \\ &\leq 2(1 - M_{T_{n-1}}^k). \end{aligned}$$

Therefore we have that

$$\begin{aligned} & \mathbb{P} \left(\tilde{R} = n | \mathfrak{F}_0 \right) \\ &= \mathbb{P} \left(\tilde{R} > n - 1, \text{Another song is played at } T_n | \mathfrak{F}_0 \right) \\ &\leq \mathbb{P} \left(\text{Another song is played at } T_n | \tilde{R} > n - 1, \mathfrak{F}_0 \right) \\ &\leq \mathbb{E} \left(2(1 - M_{T_{n-1}}^k) | \tilde{R} > n - 1, \mathfrak{F}_0 \right) \\ &= 2 \left(1 - \frac{2\eta}{N} \right)^{n-1} (1 - M_0^k). \end{aligned}$$

For the first inequality here we used the simple bound

$$\mathbb{P}(A \cap B) \leq \mathbb{P}(A|B).$$

This allows us to calculate that

$$\begin{aligned} \mathbb{E} \left(\tilde{R} 1_{\tilde{R} < \infty} | \mathfrak{F}_0 \right) &= \sum_{n \geq 0} n \mathbb{P} \left(\tilde{R} = n | \mathfrak{F}_0 \right) \\ &\leq \sum_{n \geq 0} n 2 \left(1 - \frac{2\eta}{N} \right)^{n-1} (1 - M_0^k) \\ &= 2(1 - M_0^k) \sum_{n \geq 0} n \left(1 - \frac{2\eta}{N} \right)^{n-1} \\ &\leq \frac{\eta}{2N} \frac{N^2}{4\eta^2} \\ &= \frac{N}{8\eta}, \end{aligned}$$

using our assumption that $M_0^k \geq 1 - \frac{\eta}{2N}$ and the Taylor series expansion

$$\sum_{n \geq 0} nx^{n-1} = \frac{1}{(1-x)^2},$$

for $|x| < 1$.

Our result then follows since meetings occur independently at rate $\frac{1}{N}$ and so

$$\mathbb{E}(R1_{R < \infty} | \mathfrak{F}_0) = \frac{1}{N} \mathbb{E}(\tilde{R}1_{\tilde{R} < \infty} | \mathfrak{F}_0).$$

□

We are finally prepared to prove Theorem 1.1.

Proof. We will calculate here an upper bound for

$$m = \max_{X_0} \mathbb{E}_{X_0} T$$

i.e. the upper bound over all initial configurations X_0 .

Let τ be the stopping time from Equation (6), i.e. the first time that some song k has $M_t^k \geq 1 - \frac{\eta}{2N}$ and let K be that song. Note that this defines K uniquely as $1 - \frac{\eta}{2N} \geq \frac{1}{2}$. Let R be stopping time (as above) defined by

$$R = \inf\{t \geq \tau \mid \text{some song other than } K \text{ is played}\}.$$

We first recall from Proposition 3.3 that at time τ , we have

$$\mathbb{P}_{X_\tau}(T_{\text{fix}} = 0) \geq \frac{1}{2}.$$

Also, at time τ , if T_{fix} has not yet occurred, then some song other than K will play again and so $R < \infty$.

Combining Proposition 3.1 and Proposition 3.2 we have that there exists a constant $C(\eta)$ so that from any initial configuration X_0

$$\mathbb{E}_{X_0} \tau \leq C(\eta) \frac{\ln(\sigma)N}{\lambda}.$$

We then have for any initial X_0 :

$$\begin{aligned} \mathbb{E}_{X_0} T &= \mathbb{E}_{X_0} \mathbb{E}((T_{\text{fix}} - \tau) + \tau | \mathfrak{F}_\tau) \\ &= \mathbb{E}_{X_0} \tau + \mathbb{E}_{X_0} \mathbb{E}_{X_\tau} T_{\text{fix}} \\ &= \mathbb{E}_{X_0} \tau + \mathbb{E}_{X_0} \mathbb{E}_{X_\tau} T_{\text{fix}} 1_{T_{\text{fix}} > 0} \\ &= \mathbb{E}_{X_0} \tau + \mathbb{E}_{X_0} \mathbb{E}_{X_\tau} ((T_{\text{fix}} - R)1_{R < \infty} + R1_{R < \infty}) \\ &= \mathbb{E}_{X_0} \tau + \mathbb{E} \mathbb{E}_{X_\tau} R 1_{R < \infty} + \mathbb{E} \mathbb{E}((T_{\text{fix}} - R)1_{R < \infty} | R) \\ &= \mathbb{E}_{X_0} \tau + \frac{1}{8\eta} + \mathbb{E}(1_{R < \infty} \mathbb{E}_{X_R} T) \\ &\leq C(\eta) \frac{\ln(\sigma)N}{\lambda} + \frac{1}{8\eta} + \mathbb{E}(1_{R < \infty} m) \\ &\leq 2C(\eta) \frac{\ln(\sigma)N}{\lambda} + \frac{1}{2} \max_{x_0} \mathbb{E}_{x_0} T_{\text{fix}}. \end{aligned}$$

Here the $\frac{1}{8\eta}$ is clearly dominated by the first term. Therefore, we have that

$$\max_{X_0} \mathbb{E}_{X_0} T_{\text{fix}} \leq 2C(\eta) \frac{\ln(\sigma)N}{\lambda} + \frac{1}{2} \max_{X_0} \mathbb{E}_{X_0} T_{\text{fix}}$$

and so

$$\mathbb{E}_{X_0} T_{\text{fix}} \leq 4C(\eta) \frac{\ln(\sigma)N}{\lambda}$$

from which our conclusion follows. \square

4 The Interaction Parameter η

Our goal here is find the asymptotic of our bound with respect to η . Tracing through the steps of our proof of Theorem 2.1, we may actually prove the following improved bound.

Proposition 4.1. *There exists a constant C so that from any initial configuration x_0 , the first escape time S satisfies*

$$\mathbb{E}_{x_0} S \leq \frac{C}{\eta^3(1-\eta)} \frac{N}{\lambda}.$$

Then, repeating the arguments in Section 3, we may improve our bound in Theorem 1.1 on the expectation of the fixation time T_{fix} .

Theorem 4.2. *There exists a constant C so that from any initial X_0 the fixation time T_{fix} satisfies*

$$\mathbb{E} T_{\text{fix}} \leq \frac{C}{\eta^3(1-\eta)} \frac{\ln(\sigma)N}{\lambda}.$$

We conjecture that this can actually be improved to depend on η as $\frac{1}{\eta(1-\eta)}$.

5 The Complete Graph Case

As an example of a geometry in which more can be said than Theorem 1.1, we look at the complete graph K_N on N vertices. Specifically, we have uniform meeting rates between agents, that is $\nu_{ij} = \frac{1}{N-1}$ for all pairs of agents i, j . It is standard fact that the spectral gap $\lambda_{K_N} = 1$ and so Theorem 1.1 shows that the fixation time has

$$\mathbb{E} T_{\text{fix}} = O(N).$$

A simple argument will show that this order of magnitude bound is in fact tight.

5.1 A Lower Bound

Throughout this section we assume that there are at least two songs, i.e. $\sigma \geq 2$. To achieve any reasonable lower bound, we need to ignore starting conditions that are likely already at fixation by time $t = 0$. We call an initial configuration **non-trivial** if there exists at least one song k with

$$\frac{1}{2\sigma} \leq M_0^k \leq 1 - \frac{1}{2\sigma},$$

and will consider only non-trivial initial configurations. The choice of the factor of $\frac{1}{2}$ here is of course arbitrary.

Theorem 5.1. *There exists a constant $C(\eta, \sigma)$ such that for K_N started from any non-trivial initial configuration, the fixation time T_{fix} has*

$$\mathbb{E} T_{\text{fix}} \geq C(\eta, \sigma)N.$$

Proof. Recalling Theorem 2.1, first consider any one song and consider its average preference $M_t, t \geq 0$. From the proof of Proposition 2.5

$$\mathbb{E}(dM_t(1 - M_t) | \mathfrak{F}_{t-}) = -\frac{2\eta^2}{N} Q_t dt,$$

which combined with $Q \leq \frac{1}{4}$ gives that

$$M_t(1 - M_t) - M_0(1 - M_0) + \frac{\eta^2}{2N}t$$

is a sub-martingale.

By assumption, there exists at least one song k with $M_0^k(1 - M_0^k) \geq \frac{1}{4\sigma}$. Let

$$T_2 = \inf_{t \geq 0} \{M_t^k \notin \left(\frac{1}{8\sigma}, 1 - \frac{1}{8\sigma}\right)\},$$

be the first time that M_t^k leaves the interval $(\frac{1}{8\sigma}, 1 - \frac{1}{8\sigma})$. Then applying the Optional Stopping Theorem

$$\mathbb{E} M_{T_2}^k(1 - M_{T_2}^k) + \frac{\eta^2}{2N} \mathbb{E} T_2 \geq M_0(1 - M_0) \geq \frac{1}{4\sigma}.$$

At time T_2 , we have have

$$M_{T_2}^k(1 - M_{T_2}^k) \leq \frac{1}{8\sigma},$$

and so we can conclude that

$$\mathbb{E} T_2 \geq \frac{N}{4\eta^2\sigma}.$$

To complete the proof, we need only show that the fixation time T_{fix} is with high probability the same order of magnitude as T_2 .

Consider the first meeting after time T_2 , between some agents i and j . If two different songs are played at that meeting, then by definition T_{fix} must not have yet occurred. The probability that at a meeting at time t that agent i plays song k and j does not, or vis-versa, is

$$X_t^k(i)(1 - X_t^k(j)) + X_t^k(j)(1 - X_t^k(i)).$$

Therefore, on the complete graph, the probability that two different songs play at a meeting at time t is

$$\begin{aligned} & \sum_{i \neq j} \binom{N}{2}^{-1} (X_t^k(i)(1 - X_t^k(j)) + X_t^k(j)(1 - X_t^k(i))) \\ &= \sum_{i \neq j} \frac{X_t^k(i)(1 - X_t^k(j))}{N(N-1)} \\ &= \frac{N}{N-1} M_t^k(1 - M_t^k) - \sum_i \frac{X_t^k(i)^2}{N(N-1)} \\ &\geq M_t^k(1 - M_t^k) - \frac{1}{N-1}. \end{aligned}$$

Recalling Lemma 2.2, at time T_2 we still have

$$M_{T_2}^k \in \left(\frac{1}{8\sigma} - \frac{2\eta}{N}, 1 - \frac{1}{8\sigma} + \frac{2\eta}{N} \right)$$

and so at time T_2 we have

$$M_{T_2}^k(1 - M_{T_2}^k) \geq \left(\frac{1}{8\sigma} - \frac{2\eta}{N} \right)^2$$

Thus, the probability at time T_2 that fixation has occurred is bounded by

$$\begin{aligned} \mathbb{P}_{X(T_2)}(T_{\text{fix}} \geq 0) &\geq M_{T_2}^k(1 - M_{T_2}^k) - \frac{1}{N-1} \\ &\geq \left(\frac{1}{8\sigma} - \frac{2\eta}{N} \right)^2 - \frac{1}{N-1}. \end{aligned}$$

Applying the Strong Markov property, we can conclude that

$$\begin{aligned} \mathbb{E} T_{\text{fix}} &\geq \mathbb{E} T_2 \mathbb{1}(T_{\text{fix}} \geq T_2) \\ &= \mathbb{E} T_2 \mathbb{E}(1(T_{\text{fix}} \geq T_2) | T_2) \\ &= \mathbb{E} T_2 \left(\left(\frac{1}{8\sigma} - \frac{2\eta}{N} \right)^2 - \frac{1}{N-1} \right) \\ &\geq \frac{N}{4\eta^2\sigma} \left(\left(\frac{1}{8\sigma} - \frac{2\eta}{N} \right)^2 - \frac{1}{N-1} \right) \end{aligned}$$

finishing the proof. □

6 Further Directions

We conclude by presenting a few possible further directions for research on the iPod model.

6.1 Improve the Fixation Time Bound

Heuristically, from any initial configuration the processes X_t^k mixes on a time scale of the order of the relaxation time λ^{-1} . Then, for any song k , when $x_t(i) \approx M_t$ we have $Q_t \approx M_t(1 - M_t)$ and so

$$\mathbb{E}(dM_t(1 - M_t)|F_{t-}) \approx -\frac{2\eta^2}{N}M_t(1 - M_t)dt.$$

Following through the same embedding and comparison arguments, we then find a fixation time of $O(N)$. Therefore we conjecture that for any initial configuration

$$\mathbb{E} T_{\text{fix}} = O(\lambda^{-1} + N) = O(\max(\lambda^{-1}, N)).$$

6.2 Remove the Dependence on σ

When the processes X_t^k are well mixed, i.e. when $x^k(i) \approx M^k$ again we have $Q_t \approx M_t(1 - M_t)$. Then, the σ -dimensional process $\{M_t^k\}_{1 \leq k \leq \sigma, t \geq 0}$ has a comparable covariation structure to the σ -allele Wright Fisher Diffusion.

By a well known calculation [6] the σ -allele process has an expected absorption time of $O(1)$, i.e. independent of σ . Therefore, we conjecture that by a similar embedding and comparison argument, the iPod process fixates in a time scale independent of the number of songs σ .

Combining this with our other conjectured improvements to Theorem 1.1, we conclude with the following conjectured bound for the fixation time of the iPod model.

Conjecture 1. *There exists a constant C so that for any graph \mathfrak{G} on N vertices, the fixation time T_{fix} of the iPod model on \mathfrak{G} with σ songs, started from any initial configuration, satisfies*

$$\mathbb{E} T_{\text{fix}} \leq \frac{C}{\eta(1 - \eta)} \max(N, \lambda^{-1}),$$

where λ is the spectral gap of \mathfrak{G} .

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